

Basic dynamics of split Stirling refrigerators

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ABSTRACT

The basic features of the split Stirling refrigerator, driven by a linear compressor, are described. Friction of the compressor piston and of the regenerator, and the viscous losses due to the gas flow through the regenerator matrix are taken into account. The temperature at the cold end is an input parameter. The general equations are derived which are subsequently treated in the harmonic approximation. Examples are given of application of the relations for describing optimum-performance conditions as well as the interrelationship between cooler and heat-engine operation.

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1. Introduction

In this paper we treat the split Stirling refrigerator driven by a linear compressor. From the many dissipative mechanisms in the system only the friction of the compressor piston, the friction of the regenerator, and the viscous losses due to the gas flow through the regenerator matrix will be taken into account. The temperature at the cold end is taken as an input parameter. First the general equations are derived which are subsequently treated in the harmonic approximation. In principle this limits the application of the formalism to the steady state and to low amplitudes. On the other hand it has great advantages since it leads to simple analytical formulae which leads to a clear picture of the basic operation of the machine. They allow a quick overview of the essential aspects in the design, and demonstrates fundamental parameter combinations. In addition the relations have elegant solutions which correspond to the operation as a cooler, as expected, but also as a heat engine.

In a split Stirling refrigerator the regenerator/displacer is driven by the dynamic pressure in the system. There is a finite pressure drop over the regenerator. At the same time a finite flow through the regenerator is needed to produce the cooling. The combination of the pressure drop and the volume flow is a fundamental source of dissipation. In the absence of a bouncing volume the regenerator is driven only by the pressure drop over it. In that case the split Stirling refrigerator needs a fundamentally dissipative process to generate the cooling. In this respect it resembles a pulse-tube refrigerator which needs an orifice. This type of cooler is called a *Stirling* refrigerator but the cycle differs from the fundamental Stirling cycle which is a reversible cycle between two isotherms and two isochors and has the Carnot coefficient of performance (COP). In this paper we will derive a relation for the optimum

COP of the split Stirling refrigerator without bouncing volume which is fundamentally lower than the Carnot COP. Fortunately the degradation of the performance due to the dissipation in the regenerator can be made rather small in practice.

The theoretical analysis of split Stirling cryocoolers has been the topic of various papers. The analysis starts with a paper by de Jonge [1]. It gives a phenomenological description in which the system properties such as pressure variations are related to the positions of the piston and the regenerator. The actual values of the phenomenological parameters are derived from a numerical computer program and from experiment. In this sense the phenomenological parameters are used as fitting parameters. Ackermann [2] analyzed free-piston resonant cryorefrigerators in the harmonic model emphasizing the use of the phasor formalism. Since then many papers have been published related to this topic. A number of them is given in Refs. [3–9]. For designing practical refrigerators our work results in some general rules of thumb. For more detailed design and optimization more elaborate numerical calculations are needed such as in [10] or with Sage and Regen [11].

Coolers and heat engines have much in common. As mentioned above we will show that, for certain combinations of frequency and temperatures, the split Stirling refrigerator actually operates as a free-piston Stirling heat engine. A full discussion of the system as a heat engine can be found in the book of Walker [12].

2. System description

2.1. Components

A split Stirling cryocooler consists of a compressor, usually of the dual-piston flexure-bearing type, connected to a cold finger via a split pipe. Fig. 1 represents the cooler in linear form. The exit of the compressor and the entrance of the cold finger both have a heat exchanger to ensure that the gas, shuttling between these two components through the split pipe (with negligible volume),

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Nomenclature

A	area (m ²)
D	molar flow conductance (mol/sPa)
F	force (N)
f	friction factor (N s/m)
l_s	axial length parameter (m)
H	enthalpy flow (W)
k	spring constant (N/m)
L	length (m)
m	mass (kg)
N	help parameter (Eq. (54))
$\dot{n}$	molar flow rate (mol/s)
p	pressure (Pa)
P	power (W)
Q	quality factor
$\dot{Q}$	heating or cooling power (W)
R	ideal gas constant (J/Kmol)
$\dot{S}_i$	entropy production rate (W/K)
t	time (s)
T	temperature (K)
v	velocity (m/s)
V	volume (m ³)
$\dot{V}$	volume flow rate (m ³ /s)
w	compliance factor (m ³ /Pa)
x	position (m)

x_0	regenerator position amplitude (m)
X	general dynamic system parameter
z	specific flow impedance (1/m ²)

Greek symbols

α	area ratio
γ	specific heat ratio
η	viscosity (sPa)
ν	frequency (Hz)
ξ	coefficient of performance
ϖ	help parameter (rad ² /s ²)
φ	phase angle (rad)
ω	angular frequency (rad/s)

Subscripts

0	time average
1, 2, b, c, d, e, f	spaces in the system
C	Carnot
CC	relative to Carnot
g	gas
L, H	low, high temperature
r	regenerator, resonance
Q	see Eq. (65)
x, y	in phase, out of phase

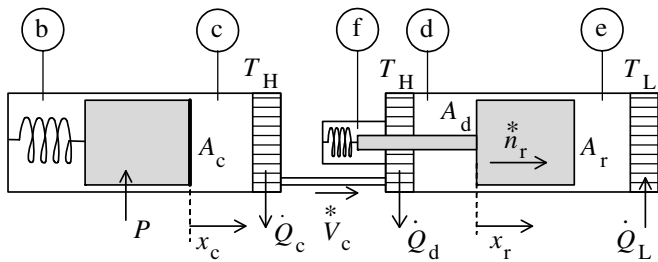


Fig. 1. Schematic diagram of the split Stirling cryocooler indicating the sign convention and the labeling of the various components and spaces.

always is at room temperature. In the cold finger the regenerator also acts as the displacer. (These functions can also be separated.) We will simply call it the regenerator. The regenerator is fixed to a guiding rod and a mechanical spring. The guiding rod can stick into a backing volume (displacer gas-spring volume). The mechanical spring can be mounted directly behind the regenerator or inside the backing volume but that makes no difference for the analysis. The motion of the regenerator is driven by the pressure difference over it (between spaces d and e) and by the pressure difference over the guiding rod (between spaces d and f). The compressor is represented as a single mass-spring system driven by an external force F applying an average power P . How the power P is generated and how the overall coefficient of performance (so including, e.g. the electrical losses) can be optimized is an interesting topic. However, in this paper, we concentrate on the cooler part and consider the power as an input parameter.

Many of the parameters describing the system are defined in Fig. 1. In mechanical terms the split Stirling cryocooler is a system of two coupled oscillators set in forced oscillation by the external force F . One oscillator is the piston with its spring and gas-spring action from space b. The other oscillator is the regenerator with its spring and the gas-spring action from the spaces d, f, and e. The cold head absorbs heat at a rate $\dot{Q}_L$ at a temperature T_L . Room

temperature will be denoted by T_H . Most of the time we will assume that $T_L < T_H$ but by the end of the paper we will see that there are special solutions of the relations which correspond to heat-engine operation, so we will also indicate what may happen if the cold end is heated to temperatures above room temperature ($T_L > T_H$).

The pressures in the system all vary around an average value $p_0 = \bar{p}$ according to $p = p_0 + \delta p$. We assume that the pressure variations are small compared to the average pressure $|\delta p| \ll p_0$. We also assume that the variations of the various volumes due to the motion of the piston and the regenerator are much smaller than the corresponding average volumes. These two assumptions allow linearization of the relations. We also assume that the pressure in each of the volumes is homogeneous and that there is no pressure drop over the split pipe so that

$$p_d = p_c. \quad (1)$$

We will meet several times the situation where the working fluid flows into a certain control volume V with volume flow $\dot{V}_1$ and flows out with volume flow $\dot{V}_2$ while the pressure p inside the volume changes with time t . If the gas in the control volume is an ideal gas $\dot{V}_1$ can be related to $\dot{V}_2$ by

$$\dot{V}_1 = \frac{V}{\gamma' p} \frac{dp}{dt} + \dot{V}_2. \quad (2)$$

This important relation is valid even if the temperature distribution in the control volume is not homogeneous. A volume flow in Eq. (2) can be replaced by a moving wall with surface area A and velocity v . In that case we have to replace V by vA . If the process in the volume is adiabatic the factor $\gamma' = \gamma = C_p/C_v$ which is the ratio of the heat capacities at constant pressure and at constant volume. If the process in the control volume is isothermal $\gamma' = 1$. If the process is in between isothermal and adiabatic some empirical value between 1 and γ can be used. In order to simplify notations we introduce the compliance factor $w_i = V_{i0}/\gamma' p_0$ (with $i = b, c, d, e$) and the molar volumes $V_H = RT_H/p_0$ and $V_L = RT_L/p_0$, with R the molar ideal-gas constant.

2.2. Compressor

For the backing volume b in the compressor holds

$$V_b = V_{b0} + A_c x_c \quad (3)$$

and for the volume c at the compression side

$$V_c = V_{c0} - A_c x_c \quad (4)$$

with x_c the displacement from the equilibrium position of the piston and A_c the area of the piston. Using the dot notation for the time derivative the rate of change of the volumes b and c are related to the piston velocity $v_c = \dot{x}_c$ by

$$\dot{V}_b = -\dot{V}_c = A_c \dot{x}_c. \quad (5)$$

It is assumed that there is no significant flow of gas to the back space. In that case with the w_i - notation introduced above

$$\delta V_b = A_c x_c = -w_b \delta p_b \quad (6)$$

and differentiating

$$\dot{V}_b = -w_b \dot{p}_b. \quad (7)$$

This relation is a special case of Eq. (2). Eq. (2), applied to the compression space c , gives

$$\dot{V}_c = -w_c \dot{p}_c - \dot{V}_c \quad (8)$$

with $\dot{V}_c$ the volume flow of the gas leaving the compressor. With Eq. (5)

$$A_c \dot{x}_c = w_c \dot{p}_c + \dot{V}_c. \quad (9)$$

2.3. Cold finger

In practice the void-volume fraction (porosity) in the regenerator may be between 30% and 70%. The hydrodynamic and thermal properties of regenerators with a nonzero void volume can be dealt with by integrating expression (88)–(92) as derived in [10]. Also numerical programs such as Sage or Regen can be used [11]. Under somewhat idealized conditions even analytical expressions, using Bessel functions, can be obtained [14]. However, these expressions are rather complicated and far from transparent. Therefore, in this treatment, we assume that the void volume in the regenerator is zero. This will limit the applicability of our approach. However, the influence of the void volume of the regenerator can be included to some extent by adding part of the void volume of the regenerator to the volumes V_{d0} and V_{e0} . In the case that no gas is stored inside the regenerator the molar flow $\dot{n}_r$ of the gas flowing through the regenerator is the same everywhere in the regenerator (what flows in flows out). In the linear (low velocity) approximation $\dot{n}_r$ is given by a Darcy type of relation

$$\dot{n}_r = -D_r p_r \quad (10)$$

with

$$p_r = p_e - p_d \quad (11)$$

and D_r the molar conductivity of the regenerator.

The volumes of spaces d , e , and f are given by

$$V_d = V_{d0} + A_d x_r \quad (12)$$

and

$$V_e = V_{e0} - A_r x_r \quad (13)$$

and

$$V_f = V_{f0} + A_f x_r. \quad (14)$$

Here A_r is the area of the regenerator which is equal to the area A_e the moving surface of space e which in turn is equal to the sum of the areas of space d A_d and of the guiding rod A_f

$$A_r = A_e = A_d + A_f \quad (15)$$

and x_r the displacement from the equilibrium position of the regenerator.

Eq. (2), applied to volume d , gives

$$\dot{V}_d = \dot{V}_c - w_d \dot{p}_d - V_H \dot{n}_r. \quad (16)$$

With Eqs. (10) and (12)

$$A_d \dot{x}_r = \dot{V}_c - w_d \dot{p}_c + V_H D_r p_r. \quad (17)$$

For volume e

$$\dot{V}_e = V_L \dot{n}_r - w_e \dot{p}_e. \quad (18)$$

From Eqs. (1), (10), (11), and (13)

$$A_r \dot{x}_r = V_L D_r p_r + w_e \dot{p}_r + w_e \dot{p}_c. \quad (19)$$

2.4. Powers

In this section we derive a relation for the power P in terms of the dynamic parameters of the system. We start with the equation of motion of the piston, with F the external force, which reads

$$m_c \ddot{x}_c = F + A_c (\delta p_b - \delta p_c) - k_c x_c - f_c \dot{x}_c. \quad (20)$$

Here k_c is the spring constant of the piston suspension (e.g. flexure bearings) and f_c the friction coefficient. With Eq. (6) we can write

$$F = m_c \ddot{x}_c + A_c \delta p_c + k'_c x_c + f_c \dot{x}_c \quad (21)$$

with

$$k'_c = \frac{A_c^2}{w_b} + k_c \quad (22)$$

the effective spring constant. The first term is due to the elasticity (gas spring) of the gas in space b . The average power, performed by the external force F on the piston, is given by

$$P = \overline{Fv_c}. \quad (23)$$

With Eq. (21)

$$P = \overline{\dot{x}_c \ddot{x}_c} m_c + \overline{\delta p_c v_c} A_c + \overline{\dot{x}_c \dot{x}_c} k'_c + \overline{f_c v_c^2}. \quad (24)$$

In this paper we will use several times the relation

$$\overline{X\dot{X}} = 0. \quad (25)$$

This holds since in the steady state

$$\overline{X \frac{dX}{dt}} = \frac{1}{2} \overline{\frac{dX^2}{dt}} = 0. \quad (26)$$

Eq. (25) confirms that, on average, no net power is needed for a periodic reversible energy conversion. Using Eq. (25) for $X = \dot{x}_c$ and $X = x_c$ respectively shows that the first and third terms in Eq. (24) are zero. So the conversions of the kinetic energy and the spring energy require no net power input. With this result and Eqs. (5) and (24) can be written as

$$P = -\overline{\delta p_c \dot{V}_c} + \overline{f_c v_c^2}. \quad (27)$$

The first term in Eq. (27) can be expressed in the dynamic parameters of the cold finger using Eq. (8)

$$P = \overline{\delta p_c (w_c \dot{p}_c + \dot{V}_c)} + \overline{f_c v_c^2}. \quad (28)$$

Using again Eq. (25), now for $X = p_c$, this reduces to

$$P = P_W + f_c \bar{v}_c^2 \quad (29)$$

where

$$P_W = \overline{\delta p_c \dot{V}_c} \quad (30)$$

is the so-called pV-work. With Eq. (16)

$$P_W = \overline{\dot{V}_d \delta p_c} + V_H \overline{\dot{n}_r \delta p_c}. \quad (31)$$

With Eqs. (1), (10), and (11)

$$P_W = \overline{\dot{V}_d \delta p_d} + V_H \overline{\dot{n}_r p_e} + D_r V_H \overline{p_r^2}. \quad (32)$$

With Eqs. (10), and (25) (for $X = p_e$) we get

$$P_W = \overline{\dot{V}_d \delta p_d} + \frac{T_H}{T_L} \overline{\dot{V}_e \delta p_e} + D_r V_H \overline{p_r^2}. \quad (33)$$

The last two terms in Eq. (33) can be recognized as due to the dissipation by the flow in the regenerator and due to the friction of the piston. The first two terms can be cast into recognizable form if we use the expression for the acceleration of the regenerator

$$m_r \ddot{x}_r = p_0 A_f + p_d A_d - p_e A_r - k'_r x_r - f_r \dot{x}_r. \quad (34)$$

Similar to the derivation of the gas spring due to space b the gas in space f leads to a gas-spring force on the regenerator, so we introduced

$$k'_r = \frac{A_f^2}{w_f} + k_r. \quad (35)$$

Multiplying relation (34) with v_r , and taking the time average and using $\overline{v_r} = 0$ and Eq. (25) for $X = x_r$ and $X = \dot{x}_r = v_r$ we end up with

$$\overline{p_d v_r A_d} = \overline{p_e v_r A_r} + f_r \overline{v_r^2} \quad (36)$$

or

$$\overline{p_d \dot{V}_d} = -\overline{p_e \dot{V}_e} + f_r \overline{v_r^2}. \quad (37)$$

Substitution in Eq. (33) gives

$$P_W = \left(\frac{T_H}{T_L} - 1 \right) \overline{\dot{V}_e \delta p_e} + D_r V_H \overline{p_r^2} + f_r \overline{v_r^2}. \quad (38)$$

The cooling power $\dot{Q}_L$ we derive from the first law applied to the space e

$$0 = \dot{Q}_L + \overline{\dot{H}_{ge}} - \overline{\dot{V}_e \delta p_e}. \quad (39)$$

The last term in the right-hand side of Eq. (39) is the pV-work. The second term is the net enthalpy flow of the working fluid flowing through the regenerator. If it can be arranged that the fluid enters and leaves space e with a constant temperature (T_L) we have

$$\overline{\dot{H}_{ge}} = 0 \quad (40)$$

so the cooling power can be expressed as

$$\dot{Q}_L = \overline{\dot{V}_e \delta p_e}. \quad (41)$$

Substitution of Eq. (41) in (38) gives

$$P_W = \left(\frac{T_H}{T_L} - 1 \right) \dot{Q}_L + D_r V_H \overline{p_r^2} + f_r \overline{v_r^2}. \quad (42)$$

Eq. (42) with (29) is a special form of the general relation [13]

$$P = \left(\frac{T_H}{T_L} - 1 \right) \dot{Q}_L + T_H \dot{S}_i. \quad (43)$$

The first term is the well-known Carnot contribution. The second term is the dissipated energy with $\dot{S}_i$ is the entropy production due to irreversible processes in the system. In our case

$$T_H \dot{S}_i = D_r V_H \overline{p_r^2} + f_r \overline{v_r^2} + f_c \overline{v_c^2}. \quad (44)$$

In fact Eq. (42) could have been written down immediately based on the first and second laws of thermodynamics and without the rather lengthy derivation given here. The agreement between Eqs. (42) and (43) is an important consistency check.

2.5. Summary

We will now summarize the results obtained so far. The average power P is given by Eq. (23) or Eq. (42) and the cooling power by (41). The force F is given by Eq. (21). The molar flow through the regenerator is given by Eq. (10). Writing $w_{cd} = w_c + w_d$, we get from Eqs. (9) and (17)

$$A_d \dot{x}_r = A_c \dot{x}_c - w_{cd} \dot{p}_c + V_H D_r \dot{p}_r. \quad (45)$$

Reminding Eq. (19)

$$A_r \dot{x}_r = V_L D_r \dot{p}_r + w_e \dot{p}_c + w_e \dot{p}_r. \quad (46)$$

The equation for the motion of the regenerator is given by Eq. (34). Using Eqs. (1) and (15)

$$A_r \dot{p}_r = -m_r \ddot{x}_r - A_f \delta p_c - k'_r x_r - f_r \dot{x}_r. \quad (47)$$

So we have the three Eqs. (45)–(47) for the position of the compressor piston x_c , the pressure drop over the regenerator p_r , and the pressure in the compression space p_c . In the next section these three relations will be solved within the powerful framework of the harmonic approximation.

3. Harmonic approximation

In this section we will first derive the general relations from the relations given above. Next we consider a situation which we will call the basic situation which demonstrates the properties of the split Stirling cryocooler in its purest form assuming that the area of the guiding rod is zero, $A_f = 0$ and the friction coefficients of the piston and of the regenerator are zero ($f_c = 0$ and $f_r = 0$).

3.1. General relations

We will now derive the in-phase and out-of-phase (lower index x and y , respectively) components of the various dynamic parameters describing the system. All variables will be decomposed in their harmonic components, e.g. the pressure drop p_r over the regenerator is written as

$$p_r = p_{rx} \cos \omega t + p_{ry} \sin \omega t \quad (48)$$

with $\omega = 2\pi\nu$ the angular frequency and ν the frequency. We will define the phases of all dynamic parameters with respect to the position of the regenerator by substituting

$$x_r = x_0 \cos \omega t \quad (49)$$

with x_0 the amplitude of the position of the regenerator. So the in-phase and out-of-phase components of the regenerator position are given by $x_{rx} = x_0$ and $x_{ry} = 0$, respectively. The velocity components of the regenerator are $v_{rx} = 0$ and $v_{ry} = -\omega x_0$.

Equating the in-phase harmonic components in Eq. (47) results in

$$A_r p_{rx} = -\varpi_r x_0 - A_f p_{cx}, \quad (50)$$

with the notation $m_r \omega_r^2 = k'_r$ and $\varpi_r = m_r (\omega_r^2 - \omega^2)$. Equating the out-of-phase harmonic components in Eq. (47) results in

$$A_r p_{ry} = -A_f p_{cy} + f_r \omega x_0. \quad (51)$$

The components of the compressor pressure can be obtained from Eq. (46) resulting in

$$-A_r \omega x_0 = V_L D_r p_{ry} - w_e \omega p_{cx} - w_e \omega p_{rx} \quad (52)$$

and

$$0 = V_L D_r p_{rx} + w_e \omega p_{cy} + w_e \omega p_{ry}. \quad (53)$$

Eqs. (50)–(53) form a set of four linear equations for p_{rx} , p_{ry} , p_{cx} , and p_{cy} . With the notation

$$N = (V_L D_r A_f)^2 + (w_e \omega A_d)^2, \quad (54)$$

and using Eq. (15), the components of the pressure drop over the regenerator are given by

$$N p_{rx} = -w_e \omega^2 [(A_f A_r + w_e \omega_r) A_d + A_f V_L D_r f_r] x_0 \quad (55)$$

and

$$N p_{ry} = \omega [(w_e \omega)^2 f_r A_d - V_L D_r A_f (A_f A_r + w_e \omega_r)] x_0. \quad (56)$$

The amplitude of the pressure drop over the regenerator satisfies

$$p_{rA}^2 = \frac{(A_f A_r + \omega_r w_e)^2 + (\omega f_r w_e)^2}{(V_L D_r A_f)^2 + (w_e \omega A_d)^2} \omega^2 x_0^2. \quad (57)$$

The lower index A is used to denote the amplitudes.

The components of p_c are given by

$$N p_{cx} = [w_e \omega^2 (A_d (A_r^2 + w_e \omega_r) + A_f V_L D_r f_r) - V_L^2 D_r \omega_r A_f] x_0 \quad (58)$$

and

$$N p_{cy} = \omega [V_L D_r A_f (A_f A_r + w_e \omega_r) + V_L^2 D_r A_f f_r - (w_e \omega)^2 f_r A_d] x_0. \quad (59)$$

The relations for the position of the piston can be obtained from Eq. (45). This results in

$$A_c \omega x_{cx} = A_d \omega x_0 + w_{cd} \omega p_{cx} + V_H D_r p_{ry} \quad (60)$$

and

$$A_c \omega x_{cy} = w_{cd} \omega p_{cy} - V_H D_r p_{rx}. \quad (61)$$

The components of the force can be obtained from Eq. (21)

$$F_{x,y} = [m_c a_c + A_c p_c + k'_c x_c + f_c v_{c,y}]. \quad (62)$$

The cooling power can be obtained with Eq. (41). Since $v_{rx} = 0$ the cooling power is

$$\dot{Q}_L = -\frac{1}{2} A_r (p_{cy} + p_{ry}) v_{ry}. \quad (63)$$

Substitution of the equations given above

$$\dot{Q}_L = \frac{1}{2} A_r V_L D_r \frac{A_d (\omega_r w_e + A_r A_f) + A_f f_r V_L D_r}{(V_L D_r A_f)^2 + (w_e \omega A_d)^2} \omega^2 x_0^2. \quad (64)$$

Eq. (64) shows that the cooling power is zero at a transition angular frequency ω_Q given by

$$\omega_Q^2 = \omega_r^2 + \frac{A_f}{m_r w_e} \left(\frac{V_L D_r f_r}{A_d} + A_r \right). \quad (65)$$

The power can be expressed in harmonic components of the force and the piston velocity as

$$P = \frac{1}{2} (F_x v_{cx} + F_y v_{cy}) \quad (66)$$

using Eqs. (62), (60), and (61). An other way to determine P is to use Eqs. (29) and (42)

$$P = \left(\frac{T_H}{T_L} - 1 \right) \dot{Q}_L + \frac{1}{2} V_H D_r p_{rA}^2 + \frac{1}{2} f_r v_{rA}^2 + \frac{1}{2} f_c v_{cA}^2. \quad (67)$$

As usual the COP is defined as

$$\xi = \frac{\dot{Q}_L}{P}. \quad (68)$$

The Carnot COP is equal to $\xi_c = T_L / (T_H - T_L)$. In order to quantize the performance of a cooler it is more appropriate to relate the COP to the Carnot COP by defining

$$\xi_{cc} = \frac{\xi}{\xi_c}. \quad (69)$$

4. Simplified cases

In this Section we will disregard the friction in the piston so that $P = P_w$. Furthermore we consider two important simplified cases: the first one is the case where there is no guiding rod. The driving force for the motion of the regenerator is the pressure drop p_r over it. The second special case is where the flow resistance of the regenerator is zero. The motion of the regenerator is due to the pressure difference over the guiding rod which has a nonzero cross section in this case.

4.1. No guiding rod

In this case $A_f = 0$. Hence $A_d = A_r$ and $k'_r = k_r$. With Eqs. (54) and (57) we get for the amplitude of the pressure drop over the regenerator

$$p_{rA} = \frac{\sqrt{\omega_r^2 + (\omega f_r)^2}}{A_r} x_0. \quad (70)$$

Note that the pressure drop over the regenerator does not depend on the molar flow conductance D_r of the regenerator. With Eq. (64) we get for the cooling power

$$\dot{Q}_L = \frac{1}{2} \frac{V_L D_r \omega_r}{w_e} x_0^2. \quad (71)$$

The cooling power is zero for $\omega_r = 0$ so if the applied frequency is equal to the mechanical resonance frequency of the regenerator. With Eqs. (67), (70) and (71) we get

$$P = \left(\frac{T_H}{T_L} - 1 + \frac{T_H}{T_L} w_e \frac{\omega_r^2 + (\omega f_r)^2}{\omega_r A_r^2} + \frac{w_e f_r \omega^2}{V_L D_r \omega_r} \right) \dot{Q}_L. \quad (72)$$

Since for $\dot{Q}_L$ we need $\omega_r > 0$. We see from Eq. (72) that the COP is reduced due to the flow resistance even if the flow resistance of the regenerator is minimized ($D_r \rightarrow \infty$).

4.2. Basic case

Now we consider a system in which the area of the guiding rod is zero, $A_f = 0$ and that the friction coefficients of the piston and of the regenerator both are zero ($f_c = 0$ and $f_r = 0$). In this situation the operation of the cooler is shown in its purest form. Therefore we call this the *Basic situation*. Now the relation (70) simplifies to

$$p_{rA} = \frac{\omega_r}{A_r} x_0. \quad (73)$$

Eq. (71) remains unchanged. The relation for the power (72) reduces to

$$P = \left(\frac{T_H}{T_L} - 1 + \frac{T_H}{T_L} \frac{w_e \omega_r}{A_r^2} \right) \dot{Q}_L. \quad (74)$$

With Eqs. (73) and (68) the COP is given by

$$\xi = \frac{T_L}{T_H - T_L}. \quad (75)$$

With $\omega_r = m_r (\omega_i^2 - \omega^2)$ and $w_e = V_{e0} / \gamma p_0$ the effective temperature T'_H is given by

$$T'_H = T_H + \frac{T_H V_{e0}}{\gamma p_0 A_r^2 m_r} (\omega_i^2 - \omega^2). \quad (76)$$

We see that in the basic case both the power P and the cooling power $\dot{Q}_L$ are proportional to the flow conductance D_r of the

regenerator. As a result the COP is independent of the flow conductance. Keeping in mind that, due to Eq. (71), for a cooler $\omega < \omega'_r$ the effective temperature $T'_H > T_H$ so $\xi < \xi_C$. The reduction of the efficiency is due to the viscous losses in the regenerator. The losses can be minimized by keeping the regenerator as light as possible (small m_r), have a small volume of the expansions space V_{e0} , and a large regenerator area A_r . Finally it is advantageous to have ω as close as possible to ω'_r . However, ω cannot be equal to ω'_r since that would lead to an infinite oscillation amplitude x_0 .

4.3. Large flow conductance

Now we turn to the case where the flow conductance of the regenerator is very large ($D_r \rightarrow \infty$) but the friction factor of the regenerator is nonzero and the motion of the regenerator is driven by the guiding rod ($A_f \neq 0$). With Eqs. (54) and (57) we get for the amplitude of the pressure drop over the regenerator

$$p_{rA\infty} = \frac{\sqrt{(A_f A_r + \omega_f \omega_e)^2 + (\omega_f \omega_e)^2}}{V_L D_r A_f} \omega x_0. \quad (77)$$

For the cooling power we get from Eq. (64) in the limit $D_r \rightarrow \infty$

$$\dot{Q}_{L\infty} = \frac{A_r}{2A_f} f_r \omega^2 x_0^2. \quad (78)$$

In the expression of the power, Eq. (67) using (77), the term due to the flow in the regenerator disappears. What remains is

$$P_\infty = \left(\frac{T_H}{T_L} - 1 + \alpha_f \right) \dot{Q}_L. \quad (79)$$

with α_f defined by $A_f = \alpha_f A_r$.

5. Results

The relations derived so far contain a wealth of phenomena. Here we will highlight some of them. As an example we treat the system mainly with the parameters given in Table 1 which correspond with a small split Stirling refrigerator. The spring constants of the piston and the regenerator are calculated from the resonance frequencies ν with $k = m(2\pi\nu)^2$ and the friction factors from the quality factors Q by $f = 2\pi\nu m/Q$.

Table 1
Input values of the system parameters, characterizing the standard situation

System parameter	Symbol	Value
Average backing volume	V_{b0}	200 cm ³
Piston diameter	D_c	15 mm
Regenerator diameter	D_r	9 mm
Length of regenerator	L_r	4 cm
Regenerator displacement amplitude	x_0	1 mm
Length of space d	L_{d0}	10 mm
Length of expansion space e	L_{e0}	10 mm
Applied power	P	60 W
Piston/spring resonating frequency	ν_c	50 Hz
Regenerator/spring resonating frequency	ν_r	70 Hz
Applied cooling power	$\dot{Q}_L$	5 W
Low temperature	T_L	60 K
Room temperature	T_H	300 K
Average pressure	p_0	3 MPa
Piston mass	m_c	100 g
Regenerator mass	m_r	6.5 g
Guiding rod area ratio	α_r	0.25
Specific heat ratio	γ	1.67
Specific flow resistivity	z_r	0.36 μm^{-2}
Viscosity	η	20 $\mu\text{Pa s}$
Quality factor compressor	Q_c	2
Quality factor regenerator	Q_r	1

5.1. Basic system

We start with a description of the results for the *Basic situation* in which there is no guiding rod ($A_f = 0$) and the friction factors of the piston and the regenerator are zero ($f_c = f_r = 0$).

Fig. 2 is a plot of the power and cooling power as functions of the frequency according to Eqs. (74) and (71). We see that for ν below the resonance frequency ν_r of the regenerator (here chosen as 70 Hz) the system is a cooler (both $\dot{Q}_L > 0$ and $P > 0$). Fig. 3 is a plot of ξ_{CC} the COP related to the Carnot COP, derived from Eqs. (69) and (75). We see that $\xi_{CC} < 1$ since there is a fundamental source of dissipation in the regenerator. However, for the parameters chosen in this example, the degradation in the performance is only a few percent.

The phase φ_c of the piston position is given by $\tan \varphi_c = -x_{cy}/x_{cx}$ (please note the minus sign). As the phase of the regenerator position is zero by definition (Eq. (49)) the phase φ_c is also the phase difference between these two moving components. Fig. 4 shows that $\varphi_c < 0$ for $\nu < \nu_r$ and changes sign at the resonance frequency of the regenerator. A negative φ_c means that the compression takes

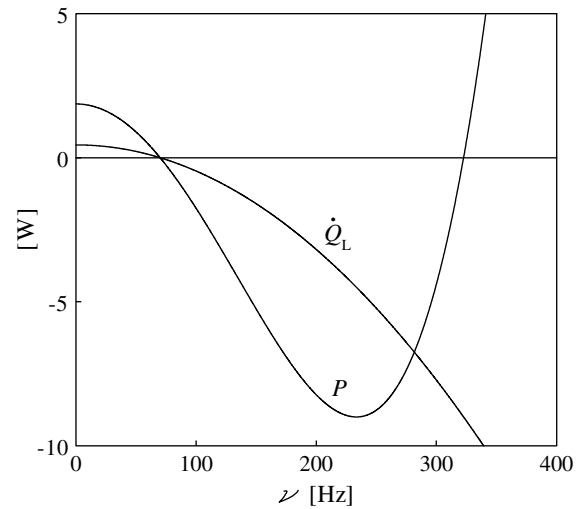


Fig. 2. Power P and applied heating power $\dot{Q}_L$ according to Eqs. (74) and (71) respectively for $x_0 = 1$ mm and $T_L = 60$ K.

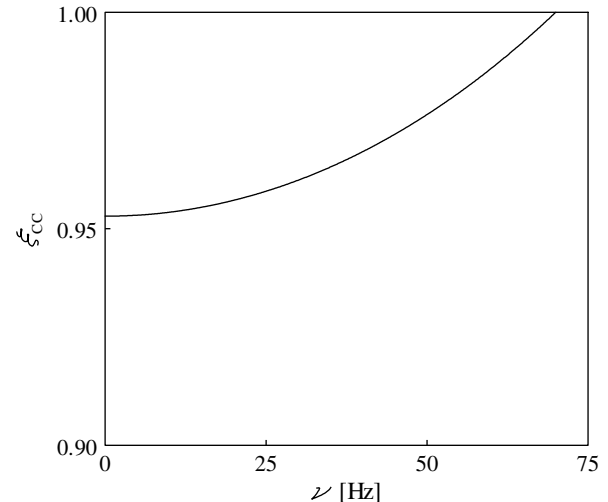


Fig. 3. Plot of ξ_{CC} , the COP relative to Carnot, as function of the frequency. Due to the dissipation in the regenerator it is always less than 1.

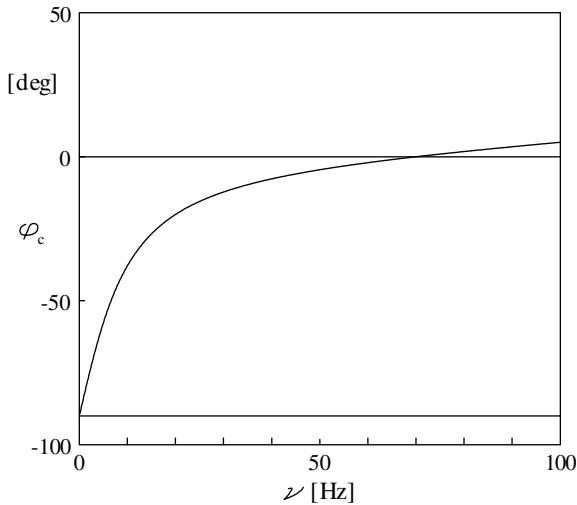


Fig. 4. Phase of the piston displacement as a function of the frequency for the basic situation.

place when the fluid is in the warm regions in the system and expansion when it is in the expansion space e , as it should for proper cooler operation.

Coming back to Fig. 2 we see that, at intermediate frequencies (70–321 Hz), both $\dot{Q}_L < 0$ and $P > 0$. This means that heat is extracted at the cold end and work is recovered at room temperature. In other words the system works as a heat engine between 300 K and the low temperature $T_L = 60$ K. For stationary operation the 60 K point has to be cooled somehow. For $\nu > 321$ Hz $\dot{Q}_L < 0$ and $P > 0$ the dissipation in the regenerator is so high that heat has to be applied at the cold end and power has to be applied to the piston.

In Fig. 5 we consider the case where $T_L > T_H$. In contrast to Fig. 2 we see that the power P and the applied heat $\dot{Q}_L$ differ in sign over the whole frequency range. At low frequencies the system works as heat engine; in the high-frequency range as a heat pump.

In this subsection we assumed that the friction of the piston and the regenerator was exactly zero. It is interesting to see what happens if the friction is nonzero, but still rather small. As an example we choose the quality factors $Q_c = Q_r = 20$. Fig. 6 is a plot of the resulting power and cooling power as functions of the frequency

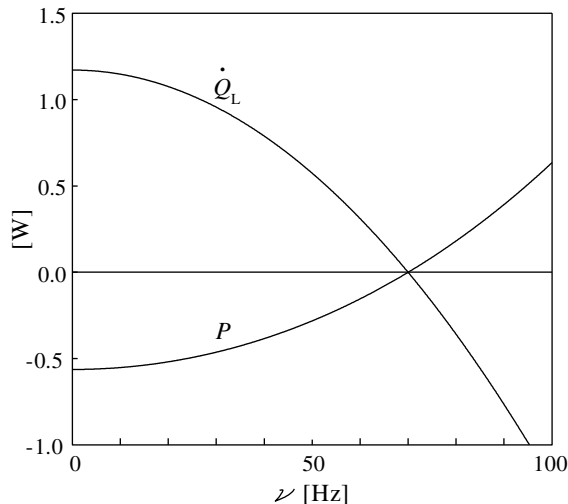


Fig. 5. Power P and applied heating power $\dot{Q}_L$ as functions of the frequency according to Eqs. (74) and (71) respectively for $x_0 = 1$ mm and $T_L = 600$ K.

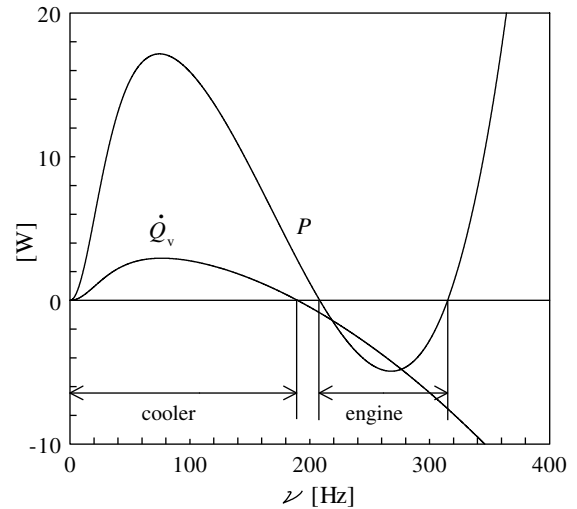


Fig. 6. Power P and applied heating power $\dot{Q}_L$ as functions of the frequency according to Eqs. (42) and (64) respectively for $x_0 = 1$ mm, $T_L = 60$ K, and $Q_c = Q_r = 20$. The frequency ranges where the system works as a cooler and as a heat engine are indicated.

according to Eqs. (67) and (64). We see that for $\nu < 190$ Hz the system is a cooler ($\dot{Q}_L > 0$ and $P > 0$). In the frequency region of 208 Hz to 315 Hz $\dot{Q}_L < 0$ and $P < 0$, so heat is extracted at the cold end and work is recovered at the hot end. In other words: the system works as a heat engine between 300 K and 60 K. The fact that the low temperature is below room temperature is impractical, but the principle remains the same. Heat-engine operation may show up in a cooler during warm up, where the heating power has been switched off. It has the tendency to keep on running. It should be noted that Fig. 6 holds for an amplitude of the regenerator of 1 mm. If the system is let to run freely as an engine it will set to a certain amplitude and frequency which depends strongly on the load. These matters have to be analyzed thoroughly. However, this is outside the scope of the paper.

5.2. General system

Now we turn our attention to the behavior of the system with parameters as given in Table 1 so with quality factors of $Q_c = 2$

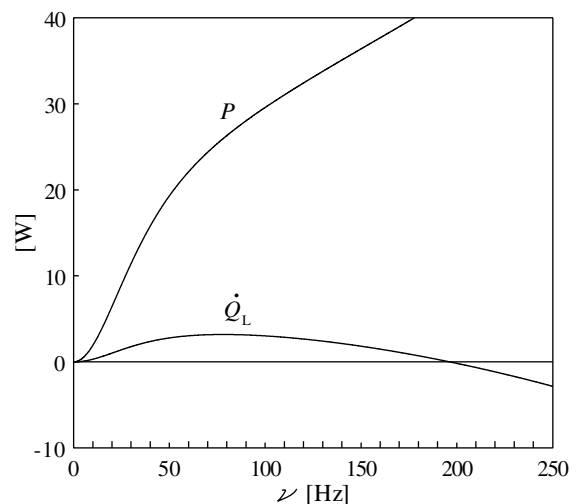


Fig. 7. Cooling power and applied power as functions of the frequency for $x_0 = 1$ mm, $T_L = 60$ K, and $Q_c = 2$ and $Q_r = 1$.

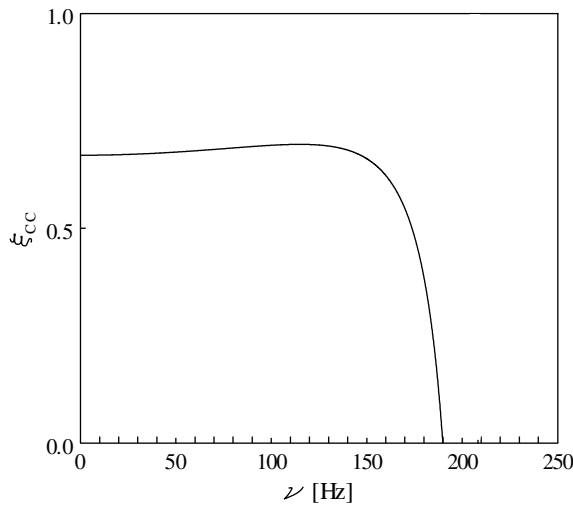


Fig. 8. Plot of the COP relative to Carnot ξ_{CC} as function of the frequency.

and $Q_r = 1$. Fig. 7 is a plot of the power and cooling power as functions of the frequency according to Eqs. (42) and (64). At frequencies above 193 Hz the system stops to work as a cooler. Fig. 8 is a plot of ξ_{CC} the COP related to Carnot. It has an optimum near 140 Hz. Naturally it is zero at the frequency where the cooling power is zero.

Fig. 9 is a plot of the ratio of the piston displacement amplitude x_{cA} to the amplitude F_A of the force. This shows the characteristic features of a resonating system with a low quality factor.

Fig. 10 is a plot of ξ_{CC} as a function of the guiding rod area ratio α_f . This graph shows that the guiding rod not only keeps the regenerator in position, but it can also be used to improve efficiency.

Fig. 11 is a plot of ξ_{CC} as a function of the specific resistivity of the regenerator material. It shows that, in general, there is an optimum value for z_r . This illustrates that some flow resistance is needed to set the regenerator in motion. If also the nonideal heat exchange between the working fluid and the matrix would be taken into account another reason for an optimum value of z_r would be involved since heat exchange and flow resistance are related.

In Fig. 7 $P > 0$ at all frequencies, so the system never operates as a heat engine due to the friction of the piston and the regenerator. The possibility of producing work is higher if the "low" temperature is increased to values above room temperature since the first

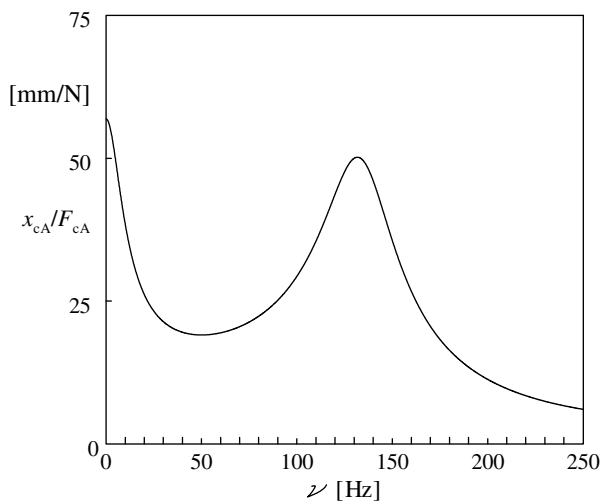


Fig. 9. Ratio x_{cA}/F_{cA} as function of the frequency.

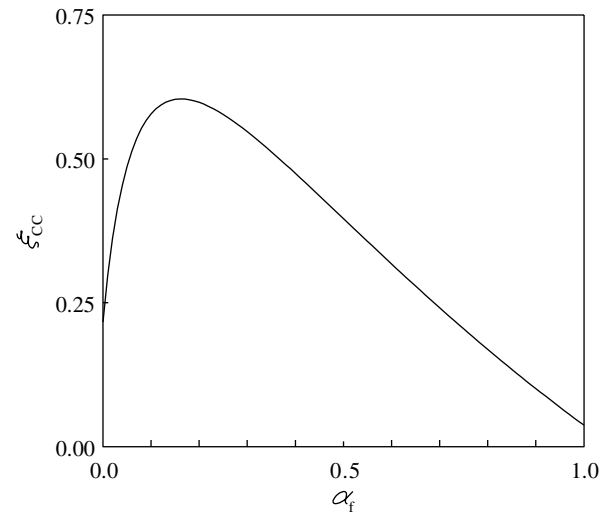


Fig. 10. Plot of ξ_{CC} as function of the relative cross section of the guiding rod.

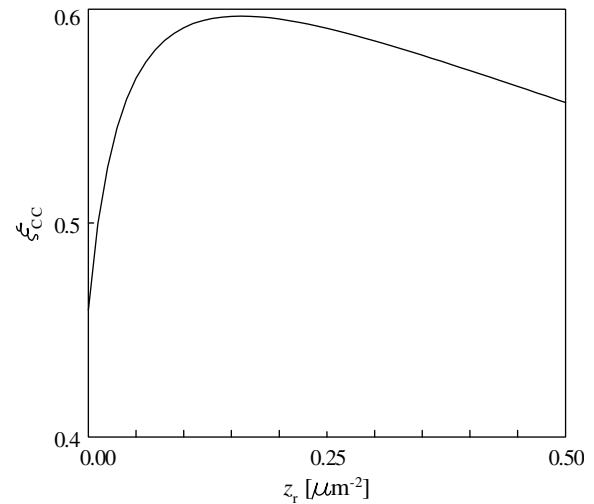


Fig. 11. Plot of the COP relative to Carnot as a function of the specific resistivity of the regenerator.

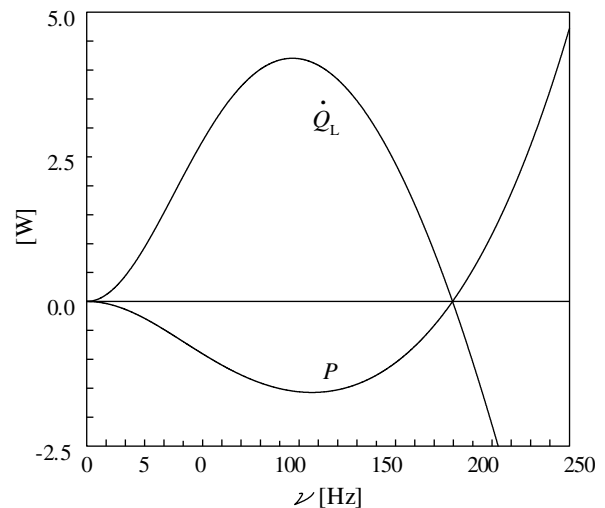


Fig. 12. Demonstration of the heat-engine characteristics if $T_L = 600$ K, in the standard case for $x_0 = 1$ mm. In the low-frequency regime heat is applied at T_L and power P is recovered.

term in Eq. (42) becomes negative. If $T_L > T_H$ the power and the applied-heat plot is as in Fig. 12. Even for the high-dissipation values of $Q_c = 2$ and $Q_r = 1$ the system works as a heat engine in the low-frequency range. In the high-frequency range the system is a heat pump.

6. Conclusion

In this paper we have derived the basic relations for the split Stirling cryocooler in the harmonic approximation assuming that the void volume in the regenerator is zero but taking the flow resistance and the friction of the piston and regenerator into account. The relations show a wealth of phenomena. Their relative simplicity allows a description of the system performance in comprehensive physical terms. Special attention is paid to the basic system in which the guiding rod is absent and the friction of the piston and the regenerator is assumed to be equal to zero. In this case the relations become even simpler and e.g. allow an analysis in term of a cooler and a heat engine.

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